Dissipation and Discontinuities

SUN Siu-wing

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Abstract

Abstract of thesis entitled Dissipation and Discontinuities:

We study some blow-up theories for the Burgers-type equations in one dimension, with different dissipative flux functions, i.e. $u_t + f(u)_x = \nu Q(u_x)_x, \quad \nu > 0$. We consider both the cases that $Q(u_x) = u_x$, which is in fact the well-known regularized Burgers equation we; and that $Q(u_x)$ is a bounded function, which is a convection-diffusion model equation. In this thesis, we impose different kinds of initial data, and investigate whether the solutions have discontinuities; to study the asymptotic behavior, and make comparisons for the theories in both cases.

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要旨

本論文旨在研究 Burgers-type 方程於加上不同的 dissipative flux 函數時(i.e. $u_t + f(u)_x = \nu Q(u_x)_x$，$\nu > 0$) 的 blow-up 理論。於本論文中，我們主要討論兩種情況，第一種是當 $Q(u_x) = u_x$ 的情況，這時的方程是最一般的 regularized Burger's 方程；而另一種情況則是對流擴散模型方程，即當 $Q(u_x)$ 為有界函數的時候。對於這些方程，我們加入不同的初值，研究在不同初值下解的連續性，觀察其漸近狀態及比較其理論。

Contents

Chapter 1

Introduction

Consider the scalar Burgers-type equation

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f(u(x,t)) = 0, \qquad u \in \mathbb{R}, \tag{1.1}$$

with imposed smooth initial datum

$$u(x, t = 0) = u_0(x). \tag{1.2}$$

It is well-known, see [14, 11, 16], that this problem has no classical solution in general. So mathematicians try to deal with the weak solutions of equation (1.1), which are realizable as the limits of small viscosity solutions of some regularized Burgers-type equations. Most often, they consider a standard regularized "viscous equation",

$$u_t + f(u)_x = \varepsilon u_{xx}, \qquad \varepsilon > 0. \tag{1.3}$$

It seems to be reasonable. Since there is a classical theory showing that the existence of the classical solution of the "viscous equation" (1.3)with the initial data (1.2), which is sufficiently smooth.

However, physically measurable viscous approximations maybe more complicated. So, noting that this "viscous equation" is not the best in [15], Tadmor E. treated more general "viscous equation",

$$u_t^\nu + f(u^\nu)_x = \nu Q(u_x^\nu)_x, \qquad \nu > 0, \tag{1.4}$$

with $Q(s)$ a monotonic increasing function, $u^\nu(x,t)$ converges to the entropy solution of (1.1), with the initial data (1.2). This method however provides us with a better estimate on the L^p-norm, for $2 \le p < \infty$, in the rate of convergence. For details, please refer to [15].

In this thesis, we mainly study the asymptotic behavior of the "viscous equation" (1.4) with different kinds of dissipative flux functions, $Q(u_x)$.

For the case that $Q(u_x)$ is a bounded and monotonic function, see [10, 6], we will show that the asymptotic behavior more or less depends on the initial data. When the imposed initial datum, $u_0 \in C_B^\infty$, has an amplitude small enough, i.e. $|u_0| \ll 1$, then the smooth solution exists. But, if the amplitude of the initial datum is too large, the solution will blow-up in some finite time, T. Besides, for the case that $Q(u_x)$ is a general bounded but non-monotonic function, see [6, 9], there is still no theory to study the asymptotic behavior. However, for the special case where $Q(u_x) = \frac{u_x}{1+u_x^2}$, it shown that the asymptotic behavior is similar to the case where $Q(u_x)$ is a bounded and monotonic function.

In chapter 2, we deal with the Burgers-type equation without viscosity, i.e. $u_t + f(u)_x = 0$, and investigate the solution of the Cauchy problem. However, it turns out that no matter how smooth the initial datum is, the classical solution still may not exist. In chapter 3, we deal with the equation with a standard viscosity εu_{xx}, $\varepsilon > 0$, added to the right hand side of the equation, and so the equation changes to $u_t + f(u)_x = \varepsilon u_{xx}$. The same as before, we investigate its asymptotic behaviour, to see whether the classical solution exists. Luckily, for this equation, smooth solutions exist with any smooth initial data; and so in chapter 3, we try to construct it explicitly. Moreover, with this results, we are interested to see how is the relation of the dissipative flux and the asymptotic behavior. And therefore, in chapter 4 and chapter 5, we will deal with a recently proposed convection-diffusion model, by considering the following two equations, which are in the same type, but with different dissipative fluxes. In chapter 4, we discuss the equation with monotonic dissipative flux:

$$u_t + f(u)_x = \nu Q(u_x)_x, \quad \text{where } Q'(u_x) > 0, \ \nu > 0. \tag{1.5}$$

and in chapter 5, we discuss the equation with a special non-monotonic dissipative flux:

$$u_t + f(u)_x = Q(u_x)_x, \quad \text{where } Q(u_x) = \frac{\nu u_x}{1 + u_x^2}, \ \nu > 0. \tag{1.6}$$

For both cases, we show that when the initial datum is large enough, the solution will blow-up. The proofs are given in section 4.1 and section 5.1 respectively, by showing that

$$\sup_{t \uparrow T} \{u_x(x,t)\} = \infty.$$

On the other hand, in section 4.2 and section 5.2, we show that when the initial datum is small enough, the behavior is totally different, that is, when initial data are small, the unique classical solution exists.

However, for the convection-diffusion model, the study of the asymptotic behavior is not complete; and many problems remain open. In chapter 5, we will give a belief conclusion for this thesis and state some of the open problems; for example, consider a model with periodic equation, the blow-up theory is still not provided. Although it is already shown that this is true by the numerical approach, see [6]. In fact, for the "viscous equation" (1.4), there are already many schemes and numerical results available to support our blow-up theories , see [6, 10, 9, 2, 1, 3, 5, 7, 8]. However, we will not consider any of them in this thesis. Our aim in this thesis is to investigate the asymptotic behavior of the convection-diffusion model from the theoretical point of view.

Chapter 2

Equation without viscosity

Consider (1.1), the scalar Burgers-type equation without viscosity

$$\text{i.e.} \quad u_t + f(u)_x = 0, \qquad u \in \mathbb{R},$$

with an initial datum (1.2), which is assumed to be smooth

$$\text{i.e.} \quad u(x, t = 0) = u_0(x) \in C^\infty.$$

It is well-known, see [14, 11, 16], that this problem has no classical solution. And the proof is also given in many books on conservation laws. Most of them simply complete the proof by dealing with the special case that $f(u) = \frac{u^2}{2}$. However, it is also true for the general case that with a general convex function, $f(u)$; and the proof is almost in the same manner as for the special case.

Now, let's recall the proof in the case $f(u) = \frac{u^2}{2}$.

Consider

$$u_t + uu_x = 0. \tag{2.1}$$

Differentiating (2.1) with respect to x, then putting $p(x, t) = u_x(x, t)$, we obtain

$$p_t + p^2 + up_x = 0. \tag{2.2}$$

The characteristic curves of (2.1) are defined by ,

$$\frac{dx}{dt} = u\left(x(t),t\right), \qquad x(t=0) = x_0.$$

Putting $P(t) = p(x(t),t)$, (2.2) and (1.2) can be re-written as the following ODE:

$$P'(t) + [P(t)]^2 = 0; \tag{2.3}$$

$$P(t=0) = p(x,t=0) = u_0'(x). \tag{2.4}$$

Solving (2.3) with (2.4) gives

$$P(t) = \frac{u_0'}{1+tu_0'}. \tag{2.5}$$

Therefore, by observation, we know that if there exists $x \in \mathbb{R}$ such that $u_0'(x) < 0$, then as time t tends to $-\frac{1}{u_0'}$, $u_x(x,t)$ tends to $-\infty$.

$$\text{i.e.} \quad u_x(x,t) = P(t) = \frac{u_0'}{1+tu_0'} \longrightarrow -\infty \quad \text{as } t \longrightarrow -\frac{1}{u_0'} > 0.$$

That means

$$|\partial_x u| \longrightarrow +\infty \qquad \text{as } t \longrightarrow \frac{1}{|u_0'|}.$$

So, it shows that if the initial datum is not a monotonic increasing function, no matter how smooth it is, the solution will blow-up in some finite time.

This method can also be used in the proof of the case of general convex function, $f(u)$. Now, we use the same method to deal with the problem with a convex function $f(u)$, i.e. $f''(u) > 0$. Let's consider the more general case by considering (1.1) and (1.2), and prove that the solution will blow-up in some finite time.

Similar to the proof for the special case above, first differentiating (1.1) with respect to x, and putting $p(x,t) = u_x(x,t)$, one obtains

$$p_t + f'(u)p_x + p^2 f''(u) = 0. \tag{2.6}$$

Putting $P(t) = p(x(t),t)$, then (1.1) gives

$$(2.7) \qquad P'(t) + [P(t)]^2 f''(u) = 0,$$

which is again an ODE problem, and with the initial value given as (2.4).

Solving (2.7) and (2.4), we obtain

$$(2.8) \qquad P(t) = \frac{u_0'}{1 + tf''(u)u_0'}.$$

Since $f''(u) > 0$, one may follow similar arguments as before, to show that

$$|\partial_x u| \longrightarrow +\infty \quad \text{as} \quad t \longrightarrow T = \frac{1}{|f''(u)u_0'|}.$$

So we have, it completes the proof for the blow-up theory of the general Burgers-type equation without viscosity.

To end this chapter, let's consider the following example to illustrate our result.

EXAMPLE Consider (2.1) with the initial data

$$u(x, t = 0) = x^2,$$

let

$$\frac{dx}{dt} = u(x(t), t).$$

Then (2.1) gives

$$u_t + \frac{dx}{dt} u_x = 0.$$

This implies

$$\frac{dx}{dt} = u = \text{constant}.$$

along the characteristic line.

However, the slope of the characteristic line at the initial point x_0 is equal to x_0^2 and the slope at another initial point x_1 is equal to x_1^2, suppose $x_0 < x_1 < 0$, so $x_0^2 > x_1^2$. Therefore, the two characteristic lines will intersect with each other. That means at this intersection, the two given initial values give two different values of solution. And hence it causes the formation of singularity.

Chapter 3

Equation with standard viscosity

In this chapter, we consider (1.3), the scalar Burgers-type equation with a standard viscous term; we also impose the initial data (1.2), which are also assumed to be smooth,

$$\text{i.e.} \quad \begin{cases} u_t + f(u)_x &= \varepsilon u_{xx}, \qquad \varepsilon > 0, \\ u(x, t=0) &= u_0(x) \in C^\infty. \end{cases}$$

We want to study the asymptotic behavior by proving that the classical solution exists.

3.1 Particular convective flux, $f(x) = \frac{u^2}{2}$

In this section, we follow the proof in Evans' book, see [4] and see also [12].

For simplicity, take $\varepsilon = 1$ in (1.3), and consider the equation,

$$u_t + uu_x = u_{xx}. \tag{3.1}$$

In the first step, we apply the Hopf-Cole transformation, see appendix A, to (3.1) as follows.

First integrating (3.1) with respect to x and putting

$$w(x,t) = \int_{-\infty}^{x} u(y,t)dy, \tag{3.2}$$

$$h(x) = w(x,t=0) = \int_{-\infty}^{x} u_0(y)dy; \tag{3.3}$$

we obtain

$$w_t - w_{xx} + \frac{1}{2}w_x^2 = 0, \tag{3.4}$$

together with the initial datum

$$w(x,t=0) = h(x). \tag{3.5}$$

By letting

$$v(x,t) = e^{-\frac{1}{2}w(x,t)}, \tag{3.6}$$

(3.4) and (3.5) become

$$v_t - v_{xx} = 0, \tag{3.7}$$

$$v(x,t=0) = e^{-\frac{1}{2}h(x)}. \tag{3.8}$$

Then solving (3.7) with (3.8) gives

$$v(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}-\frac{h(y)}{2}}dy. \tag{3.9}$$

using the relation (3.2), (3.6) and (3.9) gives

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty}\frac{x-y}{t}e^{-\frac{(x-y)^2}{4t}-\frac{h(y)}{2}}dy}{\displaystyle\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4t}-\frac{h(y)}{2}}dy}. \tag{3.10}$$

which is an exact solution of the problem (3.1) and (3.2).

Since we can find out the exact solution of the problem (3.1) with any smooth initial data explicitly, so we conclude that the Burger's equation with the viscous term εu_{xx} has a classical solution with any given smooth initial data.

3.2 Convex convective flux

In the previous section, it was proved that for a particular convective flux, $f(u) = \frac{u^2}{2}$, the classical solution exists for any smooth initial data. In this section, we want to ask whether there is similar behavior for the case with a general convex convective flux instead. It is found that the solution also exists. However, the proof is quite different. We cannot compute the solution explicitly by applying the Hopf-Cole transformation to transform the equation into heat equation. Since no matter how we transform the equation, it can never change into the form of a quasi-linear equation. However, refering to [16, 14, 13], it is known that although we cannot compute the smooth solution explicity, it is still true that it exists. But the proof of this behavior will not be included in this thesis. Interested readers are refered to [13].

Chapter 4

Equation with monotonic dissipative flux

In this chapter, we consider a new problem (1.5) and (1.2). However, we further assume that the initial datum is smooth,

$$\text{i.e.} \qquad \begin{cases} u_t + f(u)_x &= \nu Q(u_x)_x, \qquad \nu > 0 \\ u(x,t=0) &= u_0(x) \in C_B^\infty. \end{cases}$$

which is a recently proposed convection-diffusion model, see [6, 10]. Throughout this chapter, we further assume that:

(1) the dissipative flux, $Q(u_x)$, is a bounded and monotonic function,
i.e. $Q'(u_x) > 0$ and $|Q(u_x)| <$ constant;

(2) the convective flux, $f(u)$, is a convex function,
i.e. $f''(u) > 0$.

Under these assumptions, we show that the classical solution exists when the initial datum is small. On the other hand, the solution will blow-up in finite time when the initial datum is large.

For simplicity, also assume $\nu = 1$. And so, we will consider the following equation throughout this chapter instead of (1.5):

$$u_t + f(u)_x = Q(u_x)_x, \quad Q'(u_x) > 0. \tag{4.1}$$

Now let's consider the case with large initial data first.

4.1 Large initial data

Before we state the main theorem of this section, let's state our assumptions clearly as follows:

The flux functions are assumed to satisfy the following conditions,

(i) $f(u)$ and $Q(s)$ are smooth,

i.e. $f(u) \in C^\infty, \qquad Q(s) \in C^\infty;$

(ii) $f(u)$ is symmetric, but $Q(s)$ is skew symmetric,

i.e. $f(-u) = f(u), \qquad Q(-s) = -Q(s);$

(iii) $f(u)$ and $Q(s)$ are both strictly monotone,

i.e. $f'(u) > 0 \quad \forall u > 0, \qquad Q'(s) > 0 \quad \forall s;$

(iv) $f(u)$ is unbounded, but $Q(s)$ is bounded,

i.e. $f(u) \longrightarrow +\infty$ as $u \longrightarrow +\infty, \qquad Q(s) \longrightarrow Q_\infty < +\infty$ as $s \longrightarrow +\infty$.

For the initial data, we assume that

(a) u_0 is smooth;

(b) u_0 is skew symmetric;

(c) $\bar{u} \leq u_0 \leq 0, \ \forall x \geq 0$ where $\bar{u}$ is a "large" negative constant;

(d) $u_0 \leq \varphi(x), \quad \forall x \geq 0$ where $\varphi(x)$ is a smooth profile chosen to control the initial data u_0 to be "large".

Under the above assumptions, there is a main theorem showing that the solution of (1.5), (1.2) will blow-up in some finite time, see [6].

THEOREM 4.1 *Consider the initial value problem (1.5), (1.2) with all the hypotheses (i)–(iv) and (a)–(d) be satisfied. If it is further given that there exists a constant $u^* < 0$ such that $f(u^*) > 2Q_\infty$ and if $\varphi(x)$ is a smooth function satisfying $\varphi(R) = u^*$ for some $R > 0$, and*

$$Q\left(\varphi'\right)_x - f\left(\varphi\right)_x < 0, \qquad \forall x > 0. \tag{4.2}$$

Then there exists a finite $T > 0$ such that

$$\sup_x |u_x(x,t)| \longrightarrow +\infty, \qquad as \quad t \uparrow T. \tag{4.3}$$

Before turning to the proof of theorem 4.1, let's state and prove some useful lemmas first.

LEMMA 4.2 *Suppose that the fluxes $f(u)$ and $Q(s)$ satisfy the hypotheses (i)–(iv). Then for any $\bar{u} < 0$, there exists a constant $m > 1$ (possibly large), such that the boundary value problem*

$$f\left(\varphi\right)_x = mQ\left(\varphi'\right)_x, \tag{4.4}$$

$$\varphi(x) \longrightarrow \pm\bar{u} \quad as \quad x \longrightarrow \pm\infty; \tag{4.5}$$

has a solution.

Moreover, for all $x > 0$, we have $\varphi''(x) > 0$.

Proof Note that due to the symmetry of the boundary value problem (4.4), (4.5), $\varphi(x)$ is a skew symmetric function; and so, the problem is equivalent to the following problem:

$$f\left(\varphi\right)_x = mQ\left(\varphi'\right)_x, \quad 0 < x < +\infty, \tag{4.6}$$

$$\varphi(0) = 0, \quad \varphi(x \longrightarrow +\infty) = \bar{u}. \tag{4.7}$$

Therefore, we consider the proof of the theorem by dealing with the problem (4.6), (4.7) instead of (4.4), (4.5).

Consider (4.6). Integrating both sides with respect to x in the domain $[x, +\infty)$, we obtain

$$Q(\varphi') = \frac{f(\varphi) - f(\bar{u})}{m}. \tag{4.8}$$

Since $Q(s)$ is a monotonic function, that implies Q^{-1} exists; so integrating (4.8) with respect to x again in the domain $[0, x]$ gives

$$\int_0^{\varphi(x)} \frac{du}{Q^{-1}\left(\frac{f(u)-f(\bar{u})}{m}\right)} = x. \tag{4.9}$$

Choosing m sufficiently large, so that $\frac{f(u)-f(\bar{u})}{m}$ lies in the domain of Q^{-1} for all $u \in [\bar{u}, 0]$. Then it implies that the profile $\varphi(x)$ exists. Therefore, one obtains a profile $\varphi(x)$ that satisfies (4.9), that will be the solution of (4.6), (4.7), and so is the solution of (4.4), (4.5).

Now, we turn to the proof of the remaining part of the lemma. It is clear that

$$Q^{-1}\left(\frac{f(u) - f(\bar{u})}{m}\right) < 0, \quad \forall u \in [\bar{u}, 0].$$

Therefore, (4.9) implies that $\varphi(x) < 0$ for all $x > 0$. And by taking the derivative of (4.9) with respect to x, we also have $\varphi'(x) < 0$.

Thus, $\varphi(x)$ is a monotonic decreasing function smaller than zero. And so

$$f(\varphi)_x = f'(\varphi)\varphi'(x) > 0.$$

This, together with (4.6) and the property that $Q'(s) > 0$ implies $\varphi'' > 0$. □

Lemma 4.3 *Suppose that the hypotheses (b)–(d) and the inequality (4.2) holds. Then the solution of the problem (1.5), (1.2) is bounded as follows:*

$$\bar{u} \leq u(x,t) \leq 0, \quad \forall x > 0, \quad t < T; \tag{4.10}$$

and

$$u(R,t) \le u^*, \quad \forall t < T. \tag{4.11}$$

Proof Note that we will deal with (4.1) instead of (1.5). Suppose $u(x,t)$ is a solution of the problem (4.1), (1.2) for $x > 0$.

$$\text{i.e.} \quad \begin{cases} u_t + f(u)_x &= Q(u_x)_x \\ u(x,t=0) &= u_0(x) \in C_B^\infty. \end{cases}$$

Let $v(x,t) = -u(-x,t)$. So by the hypotheses (i)–(iv), we have

$$\begin{cases} v_t(x,t) &= -u_t(-x,t) \\ f(v(x,t))_x &= -f(u(-x,t))_x \\ Q(v_x(x,t))_x &= -Q(u_x(-x,t))_x. \end{cases}$$

Hence

$$\begin{aligned} v_t(x,t) + f(v(x,t))_x &= -u_t(-x,t) - f(u(-x,t))_x \\ &= -Q(u_x(-x,t))_x \\ &= Q(v_x(x,t))_x. \end{aligned}$$

Therefore, $v(x,t) = -u(-x,t)$ is also a solution of (4.1), (1.2). By the uniqueness theorem of (4.1), we have

$$u(x,t) \equiv -u(-x,t).$$

Setting $x = 0$, we obtain

$$u(0,t) \equiv 0.$$

And hence, by the consequence of the comparison principle for the scalar parabolic equation and the hypothesis (c)

$$\bar{u} \le u_0(x) \le 0,$$

one gets that

$$\bar{u} \le u(x,t) \le 0, \quad \forall x > 0, \ t < T;$$

and so the proof of (4.10) is completed.

Similarly, our hypothesis (d) implies that

$$\max_{x>0,\ T>t>0} u(x,t) \ \le \ \max_{x>0} \varphi(x).$$

So, the inequality (4.2) implies

$$u(x,t) \ \le \ \varphi(x), \quad \forall t < T,$$

and hence,

$$u(R,t) \ \le \ \varphi(R) \ = \ u^*, \forall t < T,$$

and the proof of (4.11) is completed. □

Proof *(Theorem 4.1)* By the left inequality of (4.10), and taking the integration with respect to x over the domain $(0,R)$, we obtain

$$\int_0^R u(x,t)dx \ \ge \ \int_0^R \bar{u}dx \ = \ R\bar{u}. \tag{4.12}$$

On the other hand, by first integrating (4.1) with respect to x over the domain $(0,R)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_0^R u(x,t)dx \\ &= f(u(R,t)) + f(u(0,t)) + Q(u_x(R,t)) - Q(u_x(0,t)). \end{aligned} \tag{4.13}$$

Now, we try to estimate the bounds for the terms in the right hand side of (4.13), in order to provide a contradiction to (4.12).

By the hypothesis (iv),

$$|Q| \ \le \ Q_\infty.$$

And hence, by the triangle inequality,

$$|Q(u_x(R,t)) - Q(u_x(0,t))| \le |Q(u_x(R,t))| + |Q(u_x(0,t))| \le 2Q_\infty.$$

Also, due to the assumption that $u(0,t) = 0$, and the hypotheses on $f(u)$, we obtain

$$f(u(0,t)) = f(0) = 0.$$

Finally, consider the bound for $f(u(R,t))$. By using the inequality (4.11), and the hypotheses on $f(u)$, we have

$$f(u(R,t)) \ge f(u^*),$$

and so, by our assumptions, we have

$$f(u(R,t)) \ge 2Q_\infty + \alpha,$$

for some $\alpha > 0$.

Hence, combining all these estimates with (4.13), we obtain

$$\begin{aligned} \frac{d}{dt}\int_0^R u(x,t)dx &\le -2Q_\infty - \alpha + 2Q_\infty \\ &= -\alpha \\ &< 0. \end{aligned}$$

However, this bound contradict with our inequality (4.12) that

$$\int_0^R u(x,t)dx \ge R\bar{u} \quad \text{and} \quad \int_0^R u_0(x)dx < 0.$$

Therefore, it completes our proof of theorem 4.1 that the solution will blow-up in some finite time T. Moreover, it also shows that the solution cannot remain regular beyond

$$T^* = -\frac{R\bar{u}}{\alpha} \quad .$$

□

To finish the case with large initial data, we state further the last proposition below:

PROPOSITION 4.4 *Suppose that the hypotheses (i)-(iv) hold. Then for any $u^* < 0$, there is a bounded skew symmetric function $\varphi(x)$ satisfying the inequality (4.2) and that $\varphi(R) = u^*$ for some $R > 0$.*

Proof By our previous result of lemma 4.2, we may take the profile $\varphi(x)$ that satisfies (4.6). Then for such a profile, we have

$$Q(\varphi')_x - f(\varphi)_x = (1-m)Q'(\varphi')\varphi''.$$

Since $m > 1$, $Q'(\varphi') > 0$ and $\varphi'' > 0$, therefore

$$Q(\varphi')_x - f(\varphi)_x < 0.$$

Take $\bar{u} = u^* - 1$, such that $\varphi(x) \longrightarrow \bar{u}$ as $x \longrightarrow +\infty$; with also that $\varphi(0) = 0$, and by the continuity of $\varphi(x)$, we have

$$\varphi(R) = u^* \quad \text{for some } R > 0.$$

The proof is then completed. □

Remark In this thesis, we deal only with the initial value problem without boundary. However, it is found that this blow-up result also holds for the corresponding Dirichlet problem in a finite interval $-L < x < L$. And the proof is similar to the case we studied here. For reference, readers may refer to the proof in [6] and appendix B.

Note that theorem 4.1 and proposition 4.4 together with the hypothesis (d) give the main result of this section that there is a large class of smooth bounded initial data leads to blow-up of the gradients in a finite time.

Until now, we have been dealing with the problem with any large initial data. In the next section, we still investigate the same set of problem, that means also dealing with the problem (1.5), (1.2); however we impose a small initial data instead of the large one, and study its blow-up behavior.

4.2 Small initial data

After the discussion of the problem with large initial data, we turn to the problem with small initial data in this section instead. Here we further assume that it is periodic or compactly supported, see [10]. In the previous section, we show that if the problem (1.5) with the fluxes satisfing hypotheses (i)–(iv), also with the initial data satisfing hypotheses (a)–(d), the solution will blow-up in some finite time. On the other hand, we also want to study the case when the initial data have small amplitude instead. So, in this section, we investigate the asymptotic behavior of its solution; however, it is found that the behavior is totally different to that with large initial data we studied in the previous section.

It is found that with the small initial data imposed, the classical solution exists, and this result can be summarized in the following theorem:

THEOREM 4.5 *Consider the problem (1.5), (1.2) with the dissipative flux function $Q(s)$ satisfy the following conditions:*

(A) $|Q(s)| \leq Q_\infty$;

(B) $Q'(s) > 0 \quad \forall s$;

(C) $Q'(s) \longrightarrow 0 \quad$ *as* $\quad s \longrightarrow \pm\infty$.

Suppose the range of $Q(s)$ be denoted by $Q : \mathbb{R} \longrightarrow [a, b]$, where $a < 0$ and $b > 0$. If further that $u_0 \in C^3$, and if it is sufficiently small that

$$\nu \left\| Q(u_0'(\cdot)) \right\|_{L^\infty} + 2 \left\| f(u_0(\cdot)) \right\|_{L^\infty} \leq \alpha < \nu \cdot \min(-a, b). \tag{4.14}$$

Then, there exists a unique global classical solution of (1.5), that is in $C^{2,1}(x,t)$.

Remark The space $C^{2,1}(x,t)$ is defined as

$$u(x,t) \in C^{2,1}(x,t) \iff u \in C^2(x) \cap C^1(t).$$

Before turning to the proof of theorem 4.5, we state here a lemma. However, in this section, what we concern is the theorem 4.5, so the proof of the following lemma is omitted here. Readers may refer to [10] to find out the details of this lemma.

LEMMA 4.6 *Consider the "viscous equation"*

$$u_t^\delta + f\left(u^\delta\right)_x = \nu Q\left(u_x^\delta\right)_x + \delta u_{xx}^\delta, \quad \delta > 0, \tag{4.15}$$

subject to $W^1(L^1)$-initial data.
Assume also that the conditions (A), (B) and (C) of theorem 4.5 hold.
Then there exists a sequence $\{\delta_n\}$ such that $\delta_n \downarrow 0$ and u^{δ_n} converges in the L^1-norm as $n \to \infty$.

Remark $W^1\left(L^1(x,t)\right)$ is defined as

$$u \in W^1(L^1(x,t)) \iff u_x, u_t \in L^1.$$

Remark Lemma 4.6 can be modified by replacing L^∞ instead of $W^1(L^1)$ for the initial data, and the result still holds.

Now, with the use of the lemma 4.6, we turn to the proof of the main theorem of this section.

Proof *(Theorem 4.5)* For simplicity, assume that $Q_\infty = 1$; and first consider the uniqueness of the solution.

(Uniqueness) Suppose that the classical solution of (1.5) exists.

Let $u^1(x,t)$ and $u^2(x,t)$ be two classical solutions of (1.5). So,

$$u_t^1 + f\left(u^1\right)_x = \nu Q\left(u_x^1\right)_x, \quad u^1(x,0) = u_0(x); \tag{4.16}$$

$$u_t^2 + f\left(u^2\right)_x = \nu Q\left(u_x^2\right)_x, \quad u^2(x,0) = u_0(x). \tag{4.17}$$

Subtracting (4.17) from (4.16), we obtain

$$\left(u^1 - u^2\right)_t + \left[f(u^1) - f(u^2)\right]_x = \nu\left[Q'(\xi)(u^1 - u^2)_x\right]_x, \tag{4.18}$$

$$u^1(x,0) - u^2(x,0) \equiv 0. \tag{4.19}$$

where ξ lies between u_x^1 and u_x^2.

Multiplying (4.18) by $sgn(u^1 - u^2)$ and integrating over x-domain give

$$\begin{aligned} &\int_x \left(u^1 - u^2\right)_t sgn(u^1 - u^2) + \int_x \left[f(u^1) - f(u^2)\right]_x sgn(u^1 - u^2) \\ &= \int_x \nu\left[Q'(\xi)\left(u^1 - u^2\right)_x\right]_x sgn(u^1 - u^2). \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{d}{dt}\left\|u^1(\cdot,t) - u^2(\cdot,t)\right\|_{L^1} \\ &= \int_x \left\{\nu\left[Q'(\xi)\left(u^1 - u^2\right)_x\right]_x sgn(u^1 - u^2) - \left[f(u^1) - f(u^2)\right]_x sgn(u^1 - u^2)\right\} \\ &\equiv I_1 + I_2. \end{aligned}$$

By the assumptions that $\nu > 0$ and $Q'(\xi) > 0$, we have

$$I_1 := \int_x \nu\left[Q'(\xi)\left(u^1 - u^2\right)_x\right]_x sgn(u^1 - u^2) \leq 0,$$

and by the assumption that $f(u^1) = 0$ and $f(u^2) = 0$ at $x = \pm\infty$, we have

$$I_2 := -\int_x \left[f(u^1) - f(u^2)\right]_x sgn(u^1 - u^2) = 0.$$

Therefore, we obtain

$$\frac{d}{dt}\left\|u^1(\cdot,t) - u^2(\cdot,t)\right\|_{L^1} \leq 0,$$

which implies

$$\left\|u^1(\cdot,t)-u^2(\cdot,t)\right\|_{L^1} \le \left\|u^1(\cdot,0)-u^2(\cdot,0)\right\|_{L^1}, \quad \forall t \ge 0.$$

However, by the initial condition (4.19), it implies

$$\left\|u^1(\cdot,t)-u^2(\cdot,t)\right\|_{L^1} = 0.$$

And hence, the proof of the uniqueness is then completed.

(Existence) Consider the regularized viscosity initial value problem (4.15) with initial data:

$$u^\delta(x,t=0) = u_0(x). \tag{4.20}$$

Obviously, (4.15) is parabolic, and so applying the maximum principle gives

$$\left|u^\delta(x,t)\right| \le \|u_0(\cdot)\|_{L^\infty}, \quad \forall t \ge 0.$$

Consider for small δ's, put

$$z := \nu Q\left(u_x^\delta\right) + \delta u_x^\delta - f\left(u^\delta\right), \tag{4.21}$$

and take the derivatives of x and t respectively, we obtain

$$z_x = u_t^\delta, \tag{4.22}$$

$$z_t = \nu Q'\left(u_x^\delta\right) z_{xx} + \delta z_{xx} - f'\left(u^\delta\right) z_x. \tag{4.23}$$

Since (4.23) is a parabolic equation in z, we can apply the maximum principle to get

$$\begin{aligned} &\left|\nu Q\left(u_x^\delta(x,t)\right) + \delta u_x^\delta(x,t) - f\left(u^\delta(x,t)\right)\right| \\ &\le \left\|\nu Q\left(u_0'(\cdot)\right) + \delta u_0'(\cdot) - f(u_0(\cdot))\right\|_{L^\infty}, \quad \forall t \ge 0. \end{aligned} \tag{4.24}$$

On the other hand, for given that $\alpha < \nu \cdot \min(-a, b)$, there exists a β such that $\alpha < \beta < \nu \cdot \min(-a, b)$, and hence

$$\alpha + \delta \left\| u_0'(\cdot) \right\|_{L^\infty} \leq \beta < \nu \cdot \min(-a, b) \quad \text{for sufficiently small } \delta. \tag{4.25}$$

It follows from (4.24) that for all $t \geq 0$,

$$\begin{aligned} \left| \nu Q\left(u_x^\delta(x,t)\right) + \delta u_x^\delta(x,t) \right| &\leq \left\| \nu Q\left(u_0'(\cdot)\right) \right\|_{L^\infty} + \left\| \delta u_0'(\cdot) \right\|_{L^\infty} + \left\| f(u_0(\cdot)) \right\|_{L^\infty} + \left| f\left(u^\delta\right) \right| \\ &\leq \left\| \nu Q\left(u_0'(\cdot)\right) \right\|_{L^\infty} + \left\| \delta u_0'(\cdot) \right\|_{L^\infty} + 2 \left\| f(u_0(\cdot)) \right\|_{L^\infty} \\ &\leq \alpha + \delta \left\| u_0'(\cdot) \right\|_{L^\infty} \\ &\leq \beta. \end{aligned}$$

By this inequality, we have

$$-\frac{\beta}{\nu} \leq Q\left(u_x^\delta(x,t)\right) + \frac{\delta}{\nu} u_x^\delta(x,t) \leq \frac{\beta}{\nu}.$$

So the monotonicity of Q implies that

$$\left| u_x^\delta(x,t) \right| \leq Q^{-1}\left(\frac{\beta}{\nu}\right),$$

and obviously, $Q^{-1}\left(\frac{\beta}{\nu}\right)$ is a constant independent of δ. And hence, we obtain the estimate on $\left| u_x^\delta \right|$.

Now, we turn to the estimates on $\left| u_t^\delta \right|$ and $\left| u_{xx}^\delta \right|$.

Differentiating the "viscous equation" (4.15) with respect to t, and put

$$w(x,t) := u_t^\delta(x,t),$$

we obtain

$$w_t + f'\left(u^\delta\right) w_x + f''\left(u^\delta\right) u_x^\delta w = \nu Q'\left(u_x^\delta\right) w_{xx} + \nu Q''\left(u_x^\delta\right) w_x u_{xx}^\delta + \delta w_{xx},$$

which is a parabolic equation. So, applying the maximum principle, and using the bound on $f''\left(u^\delta\right) u_x^\delta$, we obtain

$$\left| u_t^\delta(x,t) \right| \leq C\left(T, \left\| u_0(\cdot) \right\|_{L^\infty}, \left\| u_0'(\cdot) \right\|_{L^\infty}, \left\| u_t^\delta(\cdot, 0) \right\|_{L^\infty}\right),$$

which is a constant depending only on T, $\|u_0(\cdot)\|_{L^\infty}$, $\|u_0'(\cdot)\|_{L^\infty}$ and $\left\|u_t^\delta(\cdot,0)\right\|_{L^\infty}$.

Moreover, it follows from the "viscous equation" (4.15) that

$$u_t^\delta(x,0) = \nu Q'\left(u_0'(x)\right) u_0''(x) + \delta u_0''(x) - f'\left(u_0(x)\right) u_0'(x).$$

Due to the smoothness of the initial data, $\left\|u_t^\delta(\cdot,0)\right\|_{L^\infty}$ is bounded, and so is $\left|u_t^\delta(x,t)\right|$.

Finally, we estimate $\left|u_{xx}^\delta(x,t)\right|$. First, it is obvious that $Q'\left(u_x^\delta\right)$ is bounded away from zero, i.e. $Q'\left(u_x^\delta\right) \geq K > 0$, where K is a constant independent of δ.

It follows from (4.15) again that

$$\left|u_{xx}^\delta\right| \leq \frac{\left|u_t^\delta\right| + \left|f'\left(u^\delta\right) u_x^\delta\right|}{K\nu}, \tag{4.26}$$

which implies a uniform bound on $\left|u_{xx}^\delta\right|$.

Summarizing our results above, we obtain some bounds on $\left|u^\delta\right|$, $\left|u_x^\delta\right|$, $\left|u_t^\delta\right|$ and $\left|u_{xx}^\delta\right|$. With the uniform bounds on $\left|u^\delta\right|$, $\left|u_x^\delta\right|$ and $\left|u_t^\delta\right|$, and apply the lemma 4.6, one have the L^1-convergence of u^{δ_n}, and so have a pointwise convergence of u^{δ_n}.

In addition, there exists a subsequence that converges to u uniformly, let's denote it as u^{δ_n}. And with the above results, we still need to prove $u_{xx} \in W^1(L^2(x))$ in order to conclude that $u(x,t) \in C^{2,1}(x,t)$.

Since u_x is bounded uniformly, assume the measure of the x-domain is finite, then

$$\int_x |u_x|^2 \leq C$$

where C is a constant. So, we conclude that $u_x \in L^2(x)$.
Similarly, $u_{xx} \in L^2(x)$. And so have the L^2-bounds on u_x and u_{xx}.

In order to prove $u_{xx} \in W^1(L^2(x))$, differentiating the "viscous equation" (4.15) three times with respect to x, multiplying both sides by u_{xxx}^δ, and integrating the resulting equation over the x-domain, we obtain after using the three uniform bounds on $|u_x^\delta|$, $|u_t^\delta|$ and $|u_{xx}^\delta|$ that

$$\begin{aligned}\frac{d}{dt}\left\|u_{xxx}^\delta\right\|_{L^2}^2 &\le K_1\left\|u_{xxx}^\delta\right\|_{L^2}^2+K_2-2\delta\left\|u_{xxxx}^\delta\right\|_{L^2}^2- \\ &\quad 2\int_x Q'\left(u_x^\delta\right)\left(u_{xxxx}^\delta\right)^2+22\int_x\left|Q''\left(u_x^\delta\right)u_{xx}^\delta u_{xxx}^\delta u_{xxxx}^\delta\right|,\end{aligned} \tag{4.27}$$

where K_1 and K_2 are constants depending only on the initial data $u_0(x)$ and time T. Here, for simplicity, we also assume $\nu=1$.

Consider the last term of the right hand side of (4.27), we have

$$I := 22\int_x\left|Q''\left(u_x^\delta\right)u_{xx}^\delta u_{xxx}^\delta u_{xxxx}^\delta\right| \le 22\left\|Q''\left(u_x^\delta\right)u_{xx}^\delta\right\|_{L^\infty}\int_x\left|\left(\frac{1}{\epsilon^2}u_{xxx}^\delta\right)\left(\epsilon^2 u_{xxxx}^\delta\right)\right|,$$

where ϵ is an arbitrary non-zero number.

By the Cauchy-Schwartz inequality, and set $K_3 := \left\|Q''\left(u_x^\delta\right)u_{xx}^\delta\right\|_{L^\infty}$, we obtain

$$\begin{aligned} I &\le 22K_3\left[\frac{1}{2}\left(\frac{1}{\epsilon^2}\int_x\left(u_{xxx}^\delta\right)^2+\epsilon^2\int_x\left(u_{xxxx}^\delta\right)^2\right)\right] \\ &= 11K_3\left(\frac{1}{\epsilon^2}\left\|u_{xxx}^\delta\right\|_{L^2}^2+\epsilon^2\left\|u_{xxxx}^\delta\right\|_{L^2}^2\right).\end{aligned}$$

Then, taking ϵ small such that $K_3\epsilon^2 \le K$, we obtain from (4.27) that

$$\begin{aligned}\frac{d}{dt}\left\|u_{xxx}^\delta\right\|_{L^2}^2 &\le K_1\left\|u_{xxx}^\delta\right\|_{L^2}^2+K_2-2\delta\left\|u_{xxxx}^\delta\right\|_{L^2}^2-2\int_x Q'\left(u_x^\delta\right)\left(u_{xxxx}^\delta\right)^2+ \\ &\quad 11K_3\frac{1}{\epsilon^2}\left\|u_{xxx}^\delta\right\|_{L^2}^2+2K\left\|u_{xxxx}^\delta\right\|_{L^2}^2 \\ &\le \left(K_1+11K_3\frac{1}{\epsilon^2}\right)\left\|u_{xxx}^\delta\right\|_{L^2}^2+K_2-(2\delta+2K-2K)\left\|u_{xxxx}^\delta\right\|_{L^2}^2 \\ &\le \left(K_1+11K_3\frac{1}{\epsilon^2}\right)\left\|u_{xxx}^\delta\right\|_{L^2}^2+K_2 \\ &\le K_4\left\|u_{xxx}^\delta\right\|_{L^2}^2+K_2,\end{aligned}$$

where $K_4 := K_1+\frac{11}{\epsilon^2}K_3$, which is a constant not depending on δ. So, it yields a uniformly boundedness of $\left\|u_{xxx}^\delta(\cdot,t)\right\|_{L^2}$; and hence the desired $W^1(L^2(x))$-boundedness of u_{xx}. Therefore, by the Sobolev embedding, and the $W^1(L^2(x))$-boundedness of u_{xx}, we deduce the $C^{2,1}(x,t)$-boundedness of u.

$$\text{i.e.}\quad u_{xx}\in W^1(L^2(x)) \implies u\in C^{2,1}(x,t).$$

In order to complete the proof, it remains to show the existence of the weak solution.

Multiply the "viscous equation" (4.15) with a smooth test function, $\varphi(x,t) \in C_0^{2,1}(x,t)$, and then take the integration with respect to both the space and time domains, to obtain

$$\int_0^T \int_x \left[\varphi u_t^{\delta_n} + \varphi f'\left(u^{\delta_n}\right) u_x^{\delta_n}\right] = \int_0^T \int_x \left[\nu \varphi Q'\left(u_x^{\delta_n}\right) u_{xx}^{\delta_n} + \delta_n \varphi u_{xx}^{\delta_n}\right].$$

Using integration by parts implies

$$\int_0^T \int_x \varphi_t u^{\delta_n} + \int_0^T \int_x \varphi_x f\left(u^{\delta_n}\right) = \nu \int_0^T \int_x \varphi_x Q\left(u_x^{\delta_n}\right) - \delta_n \int_0^T \int_x \varphi_{xx} u^{\delta_n}. \tag{4.28}$$

Then taking $\delta_n \downarrow 0$ and using the uniform convergence of u^{δ_n} to u, we have

$$\int_0^T \int_x \varphi_t u^{\delta_n} + \varphi_x f\left(u^{\delta_n}\right) \xrightarrow{\delta_n \downarrow 0} \int_0^T \int_x \varphi_t u + \varphi_x f(u),$$

and

$$\delta_n \int_0^T \int_x \varphi_{xx} u^{\delta_n} \xrightarrow{\delta_n \downarrow 0} 0.$$

However, it remains to find out the limit of $\int_t^T \int_x \varphi_x Q\left(u_x^{\delta_n}\right)$. To this end, we need two more estimates, one is the L^∞-boundedness of u_{xx}^δ which is already proved above, and the other is the estimate of u_{xt}^δ.

For the estimate of u_{xt}^δ, we differentiate (4.15) with respect to both space and time variables, and then multiply both sides by $sgn\left(u_{xt}^\delta\right)$, and integrate over the x-domain to obtain

$$\begin{aligned} &\frac{d}{dt}\left\|u_{xt}^\delta(\cdot,t)\right\|_{L^1} + \int_x sgn\left(u_{xt}^\delta\right) f\left(u^\delta\right)_{xxt} \\ &= \nu \int_x sgn\left(u_{xt}^\delta\right) Q\left(u_x^\delta\right)_{xxt} + \delta \int_x sgn\left(u_{xt}^\delta\right) u_{xxxt}^\delta. \end{aligned} \tag{4.29}$$

The second term of the left hand side of (4.29) can be treated as

$$\begin{aligned}\int_x f\left(u^\delta\right)_{xxt} sgn\left(u^\delta_{xt}\right) &= \int_x \left[f'''\left(u^\delta\right)\left(u^\delta_x\right)^2 u^\delta_t + f''\left(u^\delta\right) u^\delta_{xx} u^\delta_t + f''\left(u^\delta\right) u^\delta_x u^\delta_{xt}\right] sgn\left(u^\delta_{xt}\right)\\ &\quad + \int_x \left(f'\left(u^\delta\right) u^\delta_{xt}\right)_x sgn\left(u^\delta_{xt}\right)\\ &= \int_x \left[f'''\left(u^\delta\right)\left(u^\delta_x\right)^2 u^\delta_t + f''\left(u^\delta\right) u^\delta_{xx} u^\delta_t + f''\left(u^\delta\right) u^\delta_x u^\delta_{xt}\right] sgn\left(u^\delta_{xt}\right).\end{aligned}$$

Therefore, due to the L^∞-boundedness u^δ, u^δ_x, u^δ_t and u^δ_{xx}, we have the following estimate:

$$\begin{aligned}\left|\int_x f\left(u^\delta\right)_{xxt} sgn\left(u^\delta_{xt}\right)\right| &\le \int_x \left|f'''\left(u^\delta\right)\left(u^\delta_x\right)^2 u^\delta_t + f''\left(u^\delta\right) u^\delta_{xx} u^\delta_t\right| + \int_x \left|f''\left(u^\delta\right) u^\delta_x u^\delta_{xt}\right|\\ &\le C_1 + C_2 \int_x \left|u^\delta_{xt}\right|\\ &= C_1 + C_2 \left\|u^\delta_{xt}(\cdot,t)\right\|_{L^1}.\end{aligned}$$

where C_1 and C_2 are constant depending only on the initial data $u_0(x)$ and the final time T.

Besides, obviously, the second term of the right hand side of (4.29), $\delta \int_x sgn\left(u^\delta_{xt}\right) u^\delta_{xxxt}$, is non-positive.

Finally, consider the first term of the left hand side of (4.29),

$$\begin{aligned}&\nu \int_x Q\left(u^\delta_x\right)_{xxt} sgn\left(u^\delta_{xt}\right)\\ &= \nu \int_x \left(Q'\left(u^\delta_x\right) u^\delta_{xxt}\right)_x sgn\left(u^\delta_{xt}\right) + \nu \int_x \left(Q''\left(u^\delta_x\right) u^\delta_{xx} u^\delta_{xt}\right)_x sgn\left(u^\delta_{xt}\right)\\ &= \nu \int_x \left(Q'\left(u^\delta_x\right) u^\delta_{xxt}\right)_x sgn\left(u^\delta_{xt}\right),\end{aligned}$$

which is also non-positive.

Hence, putting these bounds into (4.29), we obtain

$$\begin{aligned}\frac{d}{dt}\left\|u^\delta_{xt}(\cdot,0)\right\|_{L^1} &\le \nu \int_x \left(Q'\left(u^\delta_x\right) u^\delta_{xxt}\right)_x sgn\left(u^\delta_{xt}\right) - \int_x f\left(u^\delta\right)_{xxt} sgn\left(u^\delta_{xt}\right)\\ &\le -\int_x f\left(u^\delta\right)_{xxt} sgn\left(u^\delta_{xt}\right)\\ &\le C_2 \left\|u^\delta_{xt}(\cdot,t)\right\|_{L^1} + C_1.\end{aligned}$$

Then, by the Gronwall inequality, we obtain

$$\left\|u_{xt}^{\delta}(\cdot,t)\right\|_{L^1} \leq const_T,$$

where $const_T$ is a constant depending only on $\|u_0(\cdot)\|_{L^\infty}$, $\|u_0'(\cdot)\|_{L^\infty}$, $\|u_0''(\cdot)\|_{L^\infty}$, $\left\|u_{xt}^{\delta}(\cdot,0)\right\|_{L^\infty}$, T and the measure of the x-domain, which is also finite.

In addition, as the initial data u_0 is in C^3, $\left\|u_{xt}^{\delta}(\cdot,0)\right\|_{L^1}$ is bounded; which follows by taking the derivative of the "viscous equation" (4.15) with respect to x to get

$$u_{xt}^{\delta} = \nu Q''\left(u_x^{\delta}\right)\left(u_{xx}^{\delta}\right)^2 + \nu Q'\left(u_x^{\delta}\right)u_{xxx}^{\delta} + \delta u_{xxx}^{\delta} - f''\left(u^{\delta}\right)\left(u_x^{\delta}\right)^2 - f'\left(u^{\delta}\right)u_{xx}^{\delta}$$

Therefore, with (4.26), it deduce that $u_x^{\delta} \in W^1(L^1(x,t))$; and so, u_x^{δ} is compactly imbedded in $L^1(x,t)$. That means there exists a subsequence of u^{δ_n} such that $u_x^{\delta_n} \longrightarrow u_x$ in $L^1(x,t)$. And hence

$$\nu\int_0^T\int_x Q\left(u_x^{\delta_n}\right)\varphi_x \overset{\delta_n\downarrow 0}{\longrightarrow} \nu\int_0^T\int_x Q(u_x)\varphi_x.$$

Combining our results with (4.28), as $\delta_n \downarrow 0$, we have that

$$\int_0^T\int_x [\varphi_t u + \varphi_x f(u)] = \int_0^T\int_x [\nu\varphi_x Q(u_x) - \delta\varphi_{xx}u]$$

holds. That means, for all test function $\varphi(x,t)$, the limit function $u(x,t)$ satisfies the equation in the integral sense, and so we obtain the weak solution. Therefore, with our perivous results that $u(x,t) \in C^{2,1}(x,t)$, the classical solution exists. And hence, we conclude that when the initial data is sufficiently small, the classical solution exists for the problem (1.5) with a monotonic increasing function, $Q(s)$. □

4.3 Unbounded dissipative flux

In the previous sections of this chapter, we simply deal with a bounded monotonic dissipative flux. Also we found that for the problem (1.5), (1.2), the asymptotic behavior of solution depends mainly on the initial data; it will blow-up if the initial data has large amplitude, and the classical solution exists if the initial data has small amplitude. However, there is one more important property that we are based on, that is the boundedness of $Q(s)$. And it is found that the boundedness is necessary for the blow-up behavior we obtain in the previous two sections.

In this section, we concentrate on the case with unbounded monotonic dissipative flux, and investigate its asymptotic behavior. However this is an easy case, see [6]. Thus, we state a theorem showing that the classical solution still exists for quite general initial data.

THEOREM 4.7 **(A Priori Estimate of u_x)** *Consider the equation (1.5), with $Q(u_x)$ is a monotonic function that $Q(s) \longrightarrow \pm\infty$ as $s \longrightarrow \pm\infty$. Then $u_x(x,t)$ is uniformly bounded for all $t > 0$.*

Proof For simplicity, consider (4.1) instead of (1.5).
Let $U(x,t)$ be the primitive of $u(x,t)$,

$$\text{i.e.} \quad U(x,t) = \int_0^x u(y,t)dy.$$

Then integrate (4.1) with respect to x in the domain $(0,x)$ to get

$$\frac{d}{dt}U + f(U_x) = Q(U_{xx}), \tag{4.30}$$

and then take the derivative of t to obtain

$$U_{tt} + f'(U_x)U_{xt} = Q'(U_{xx})U_{xxt}. \tag{4.31}$$

Due to the positivity of $Q'(s)$, we observe that (4.31) is a parabolic equation of U_t. Then apply the maximum principle, and provides a uniform bound on U_t.

Similarly, one can show that u is bounded. And hence, due to the continuity of $f(u)$, it also implies the bounds on $f(u)$.

On the other hand, with the fact that $f(u) = f(U_x)$, so $f(u)$ bounded implies $f(U_x)$ is bounded. And therefore apply to (4.30), implies $Q(U_{xx})$ is also bounded.

$$\text{i.e. } |Q(U_{xx})| = |Q(u_x)| \le C,$$

where C is a constant. So, by the monotonicity and the boundedness of Q, we have $u_x(x,t)$ is uniformly bounded. And hence the proof is then completed. □

To finish this chapter, let's summarize what we have obtained. In this chapter, we have shown that for the Burgers-type equation $u_t + f(u)_x = \nu Q(u_x)_x$, $\nu > 0$, with a monotonic dissipative flux, $Q(u_x)$; the singularity of solution depends on both the boundedness of $Q(u_x)$ and the initial data:

- if $Q(u_x)$ is unbounded, the classical solution exists;
- if $Q(u_x)$ is bounded, there are two cases:
  - for smooth initial data with small amplitude, the classical solution exists;
  - for smooth initial data with large amplitude, the solution will blow-up in some finite time.

Chapter 5

Equation with non-monotonic dissipative flux

In the previous chapter, we already discuss the Burgers-type equation with bounded and monotonic dissipative flux. It is found that the classical solution exists if the imposed smooth initial data are sufficiently small; and if the smooth initial data are large enough, the solution will blow-up in some finite time. Now, in this chapter, we turn to the case with the similar type of equation, however with a non-monotonic dissipative flux instead. Unfortunately, it is hard to deal with the case of general non-monotonic dissipative flux. Therefore, in this chapter, we concentrate on the case with a special dissipative flux, $Q(u_x) = \frac{\nu u_x}{1+u_x^2}$. That is, in this chapter, we study the problem (1.6) with equation of the form

$$u_t + f(u)_x = Q(u_x)_x, \quad Q(u_x) = \frac{\nu u_x}{1+u_x^2}, \quad \nu > 0.$$

In this case, it will be shown that the asymptotic behavior of solutions is similar to that for the problem we discussed in chapter 4, in the sense that if we impose a smooth initial datum to the problem. The blow-up behavior will depend on the amplitude of the initial datum such that if the initial data is small enough, the classical solution exists; but if the initial amplitude is large enough, the solution will blow-up in some finite time, T, see [6, 9].

Here, let's also impose initial data as in (1.2),

$$\text{i.e.} \quad u(x, t = 0) \;=\; u_0(x) \in C^\infty.$$

Our main assumptions are:

(I) $f(u)$ is smooth,
i.e. $f \in C^\infty$;

(II) $f(u)$ is symmetric,
i.e. $f(-u) \;=\; f(u), \quad \forall u$;

(III) $f(u)$ is strictly monotone for all $u > 0$,
i.e. $f'(u) > 0, \quad \forall u > 0$;

(IV) $f(u)$ is unbounded,
i.e. $f(u) \longrightarrow +\infty, \quad \text{as} \quad u \longrightarrow +\infty.$

Besides, we also impose the conditions (a)–(d) in chapter 4 for the initial data. First, let's consider the case with the large initial data; however, since it is quite similar to the problem with a monotonic dissipative flux function, we shall follow the steps in chapter 4, and omit those identical steps.

5.1 Large initial data

Let's consider the problem (1.6), (1.2); with large initial data. It is found that the solution will blow-up in some finite time, see [6], and the following theorem describes clearly this effect:

THEOREM 5.1 *Consider the initial value problem (1.6), (1.2) with all the hypotheses (I)-(IV) and (a)-(d) be satisfied. If it is further given that there exists a constant $u^* < 0$ such that $f(u^*) > \nu$ and if $\varphi(x)$ is a smooth function satisfying $\varphi(R) = u^*$ for some $R > 0$ and*

$$Q(\varphi')_x - f(\varphi)_x < 0, \quad \forall x > 0. \tag{5.1}$$

Then there exists a finite $T > 0$ such that

$$\sup_x |u_x(x,t)| \longrightarrow +\infty \quad as \quad t \uparrow T.$$

Before turning to the proof of theorem 5.1, let's state and prove some useful lemma first. However, these lemmas are quite similar to those lemmas correspond to the theorem 4.1 for the case with monotonic dissipative flux.

LEMMA 5.2 *Suppose that the flux function $f(u)$ satisfies the hypotheses (I)-(IV). Then for any $\bar{u} < 0$, there exists a constant $m > 1$ (possibly large), such that the boundary value problem:*

$$f(\varphi)_x = mQ(\varphi')_x, \quad 0 < x < +\infty, \tag{5.2}$$

$$\varphi(0) = 0, \quad \varphi(x \longrightarrow +\infty) = \bar{u}, \tag{5.3}$$

has a solution.

Moreover, for all $x > 0$, we have

$$\varphi''(x) > 0 \quad and \quad -1 \leq \varphi'(x) < 0. \tag{5.4}$$

Proof It is obviously that $Q(s)$ is a non-invertible function; however, due to its property of monotonicity, consider $s \in [-1, 1]$, its inverse exists. i.e. $Q^{-1}(z)$ exists and can be defined as:

$$Q^{-1} = \frac{\nu - \sqrt{\nu^2 - 4z^2}}{2z}, \quad 0 < |z| \leq \frac{\nu}{2}.$$

By taking the derivative of $Q^{-1}(z)$ with respect to z, we obtain

$$\begin{aligned}
[Q^{-1}(z)]' &= \frac{1}{2}\left(\frac{\nu^2 - \nu\sqrt{\nu^2 - 4z^2}}{z^2\sqrt{\nu^2 - 4z^2}}\right) \\
&\geq \frac{1}{2}\left(\frac{\nu^2 - \nu\sqrt{\nu^2}}{z^2\sqrt{\nu^2 - 4z^2}}\right) \\
&= \frac{1}{2}\left(\frac{\nu^2 - \nu^2}{z^2\sqrt{\nu^2 - 4z^2}}\right) \\
&= 0.
\end{aligned}$$

Therefore, $Q^{-1}(z)$ increase monotonously. With this property, solving (5.2), (5.3) as in the proof of lemma 4.2, we obtain

$$x = \int_0^{\varphi(x)} \frac{du}{Q^{-1}\left(\frac{f(u)-f(\bar{u})}{m}\right)}. \tag{5.5}$$

Note that the integral of (5.5) is defined for sufficiently large m that

$$\left|\frac{f(u) - f(\bar{u})}{m}\right| < \frac{\nu}{2}.$$

And hence, for a suitable φ such that (5.5) holds, the first statement of lemma 5.2 is then proved. That means the problem (5.2), (5.3) has a solution φ. To complete the proof of the lemma, it remains to show (5.4),

$$\text{i.e.} \quad \varphi''(x) > 0 \quad \text{and} \quad -1 \leq \varphi'(x) < 0 \quad \forall x > 0.$$

However, following the proof of the lemma 4.2, we still obtain

$$\varphi'(x) < 0 \quad \text{and} \quad \varphi''(x) > 0.$$

and so, the first inequality of (5.4) holds.

Now consider (5.5). By taking the derivative with respect to x, we have

$$\varphi' = Q^{-1}\left(\frac{f(\varphi) - f(\bar{u})}{m}\right).$$

Therefore, for $s \in [-1, 1]$,

$$|\varphi'| = \left|Q^{-1}\left(\frac{f(\varphi) - f(\bar{u})}{m}\right)\right| \leq 1.$$

It follows that

$$-1 \leq \varphi'(x) < 0,$$

and hence, the proof of lemma 5.2 is completed. □

LEMMA 5.3 *Suppose that the hypotheses (b)–(d) in chapter 4 and the inequality (5.1) hold. Then the solution of the problem (1.6),(1.2) is bounded as follows:*

$$\bar{u} \leq u(x,t) \leq 0 \quad \forall x > 0, \quad t < T; \tag{5.6}$$

and

$$u(R,t) \leq u^*, \qquad \forall t < T. \tag{5.7}$$

Proof It is clear first that the inequality (5.6) is simply the consequence of the maximum principle for the equation (1.6).

So, we turn to the proof of the inequality (5.7). First consider (1.6) and (5.2) subtract them to obtain

$$(u-\varphi)_t + [f(u) - f(\varphi)]_x = \nu \left(\frac{u_x}{1+u_x^2} - m\frac{\varphi'}{1+(\varphi')^2} \right)_x . \tag{5.8}$$

Let

$$y(t) := \max_{x>0} (u(x,t) - \varphi(x)) = u\left(x_m(t),t\right) - \varphi(x_m(t)),$$

where $(x_m(t),t)$ is the maximum point of $y(t)$.
The right hand side of (5.8) is

$$\nu \left(\frac{u_x}{1+u_x^2} - m\frac{\varphi'}{1+(\varphi')^2} \right)_x = \nu \left(\frac{u_{xx}}{1+u_x^2} - \frac{2u_x^2 u_{xx}}{(1+u_x^2)^2} - m\frac{\varphi''}{1+(\varphi')^2} + m\frac{2(\varphi')^2\varphi''}{(1+(\varphi')^2)^2} \right).$$

However, note that, at the maximum point $(x_m(t),t)$, we have

$$u_x = \varphi' \quad \text{and} \quad u_{xx} \leq \varphi''. \tag{5.9}$$

So, at the maximum point $(x_m(t), t)$,

$$\begin{aligned}
&\nu\left(\frac{u_x}{1+u_x^2} - m\frac{\varphi'}{1+(\varphi')^2}\right)_x \\
&= \frac{\nu}{(1+\varphi')^2}\left[u_{xx}\left(1+(\varphi')^2\right) - 2(\varphi')^2 u_{xx} - m\varphi''\left(1+(\varphi')^2\right) + 2m(\varphi')^2(\varphi'')\right] \\
&= \frac{\nu}{(1+\varphi')^2}\left[u_{xx}\left(1-(\varphi')^2\right) - m(\varphi'')\left(1-(\varphi')^2\right)\right] \\
&= \nu\frac{1-(\varphi')^2}{(1+(\varphi')^2)^2}(u_{xx} - m\varphi'').
\end{aligned}$$

Recall our pervious results and assumptions that

$$\nu > 0, \quad m > 1, \quad |\varphi'| \le 1, \quad u_{xx} \le \varphi''.$$

Therefore, it implies, at the maximum point $(x_m(t), t)$,

$$\nu\left(\frac{u_x}{1+u_x^2} - m\frac{\varphi'}{1+(\varphi')^2}\right)_x \le 0.$$

And hence, we obtain

$$(u-\varphi)_t + [f(u) - f(\varphi)]_x \le 0. \tag{5.10}$$

Denote $\dot{y}(t) = \frac{d}{dt}y(t)$, so (5.10) can be re-written as

$$\dot{y} + [f'(u)u_x - f'(\varphi)\varphi'] \le 0.$$

By the mean-value theorem and the property that $u_x = \varphi'$ at the maximum point $(x_m(t), t)$, we have at the maximum point,

$$\dot{y} + \varphi'(x_m(t))f''(\xi)y \le 0, \tag{5.11}$$

where $\xi = \xi(t)$ is a mid-point between $u(x_m(t), t)$ and $\varphi(x_m(t))$.

Solving the ordinary differential inequality (5.11), one can obtains

$$y(t) \le y(0)e^{-\int_0^t \varphi'(x_m(s))f''(\xi(s))ds}.$$

Note that $y(0) \le 0$ implies $y(t) \le 0$. So putting $x = R$, we obtain the inequality (5.7). And hence, the proof of lemma 5.3 is then complete. □

With lemma (5.2) and lemma (5.3) at hand, then the proof of theorem 5.1 can be completed easily by just following the arguments of the proof of theorem 4.1. And hence, we conclude that for the Burgers-type equation (1.6), the solution will blow-up in some finite time when large initial data is imposed. Thus, the proof of theorem 5.1 is then completed.

We finish this section by stating (without proof) a parallel result as in proposition 4.4.

PROPOSITION 5.4 *Suppose that the hypotheses (I)–(IV) hold. Then for any* $u^* < 0$, *there is a bounded skew symmetric function* $\varphi(x)$ *satisfying the inequality (5.1) and that* $\varphi(R) = u^*$ *for some* $R > 0$.

The proof is completely identical to that for proposition 4.4. So details are omitted.

5.2 Small initial data

After the discussion on the equation (1.6) with large initial data in the previous section, let's turn to the case with small initial data in this section. For the solution of this problem, it is found that it is also similar to the problem we discuss in section 4.2 that the classical solution exists. Let's summarize this behavior in the following theorem:

THEOREM 5.5 **(Existence and Uniqueness)** *Consider the problem (1.6), (1.2), with the initial data,* $u_0 \in C^3$. *If the initial data satisfies*

$$(5.12) \quad \nu \left\| \frac{u_0'(\cdot)}{1 + (u_0'(\cdot))^2} \right\|_{L^\infty} + s \, \| f(u_0(\cdot)) \|_{L^\infty} \le \alpha < \frac{\nu}{2}, \quad \| u_0'(\cdot) \|_{L^\infty} < 1.$$

Then there exists a unique global classical solution $u(x,t) \in C^{2,1}(x,t)$.

The proof of this theorem is a modification of that for theorem 4.5, and we give belief description below.

In order to prove theorem 5.5, one may consider the "viscous equation"

$$u_t^\delta + f\left(u^\delta\right)_x = \nu \left(\frac{u_x^\delta}{1 + (u_x^\delta)^2} \right)_x + \delta u_{xx}^\delta, \quad \delta > 0,$$

and with the use of lemma 5.6 which is stated below, one can conclude that the unique classical solution exists for the problem (1.6), with a smooth initial data (1.2) satisfying (5.12). Now, let's state lemma 5.6 without any proof below:

LEMMA 5.6 **($W^1(L^\infty)$ A Priori Estimate)** *Let $u(x,t)$ be a classical solution of the problem (1.6), (1.2); if (5.12) is also holds for the initial data, then for all $t \geq 0$ we have $\|u_x(\cdot,t)\|_{L^\infty} \leq C$, where C is a constant.*

For the details of the proof, readers may refer to the articles "On the Burgers-type equations" written by Kurganov A. Levy D. and Rosenau P. and "Effects of a saturating dissipation in Burgers-type equations" written by Kurganov A. and Rosenau P.; see [9, 10].

We finish this chapter by summarizing the results obtained thus far:

In this chapter, we have shown that for the problem (1.6), the asymptotic behavior of the solution also depends on the smooth initial data; if the initial data have small amplitude, the classical solution exists, however, if the amplitude of the initial data is large, the solution will blow-up in some finite time, T.

Chapter 6

Comparison and conclusions

In this thesis, first, we have shown in chapter 2 that for the Burgers-type equation (1.1), which has no viscous term, it has no classical solution in general. Since if we impose a smooth initial data (1.2), no matter how smooth it is, say, in C^{∞}; if the initial data u_0 satisfies $u_0' < 0$, the solution must blow-up in some finite time, T.

However, for dealing with the equation (1.3), which consists a viscous term εu_{xx} in the right hand side of the equation, with also $\varepsilon > 0$. The behavior of solution is totally different. It is found that the classical solution exists. Furthermore, for a special Burger's equation with standard viscosity, $u_t + uu_x = \varepsilon u_{xx}$, if the initial data (1.2) is smooth enough, the classical solution can be found out explicitly by first using the Hopf-Cole transformation to transform the equation into a heat equation. And hence, it can be used to construct the weak solution to the problem (1.1).

Moreover, in this thesis, we are not only dealing with this special case, we also dealing with two different Burgers-type equations, see chapter 4 and 5, which is called a convection-diffusion model, and is proposed in [6, 10, 9]. However, for these two types of equation, the situation is not as simple as the special case (1.3). In chapter 4, we deal with the equation (1.5):

$$u_t + f(u)_x = \nu Q(u_x)_x, \quad \nu > 0,$$

where the dissipative flux is a monotonic increasing and bounded function. It is shown that the solution for this problem will blow-up in some finite time when the imposed initial data is large enough; however when the smooth initial has small amplitude, classical solution exists. And this result can be summarized by theorem 4.1, 4.5 and proposition 4.4. Finally, in chapter 5, we deal with the equation (1.6):

$$u_t + f(u)_x = \nu Q(u_x)_x, \quad Q(u_x) = \frac{u_x}{1+u_x^2}, \quad \nu > 0,$$

with the dissipative flux not monotone. It is shown that the classical solution exists if the initial data is small, and blow-up happens if the initial data is large. We summarize this behavior by theorem 5.1, 5.5 and proposition 5.4.

From these results, we can observe that for the two-types of Burgers-type equations we discuss in this thesis, the asymptotic behavior is not the same as either the Burgers-type equation without viscosity or with a standard monotonic and bounded viscosity. However, it is found that, if the initial data is small, both cases will pose a unique classical solution. Thus, if the initial data is small, the problem behaves similar to the problem with a standard viscosity. On the other hand, if the initial data is large, the solution will blow-up in some finite time. So it will be quite similar to the problem without viscosity.

This behavior can be explained by observing that when the problem without viscosity, the solution will blow-up, no matter how smooth the initial data is. However with the standard viscous term added to it, this viscous term will smooth the solution and so classical solution exists. However for the dissipative flux $Q(u_x)$ that is either bounded and $Q'(u_x) > 0$ or $Q(u_x) = \frac{u_x}{1+u_x^2}$, the situation is more complicated and interesting. It is quite surprising that the form of the viscosity plays such an important role in the behavior of solutions. Indeed, in both cases, the dissipative flux still acts as a role to smooth the solution for smooth initial data, but if the amplitude of the initial data is too large, the dissipative fluxes cannot present the formation in the solutions, so it can be compared to the Burgers equation without viscosity.

Besides, although the non-classical theories we study in chapter 4 and 5 did not provide with us a complete theory as the classical does in chapter 3. However, they are still very useful. First, as we know, the equation (1.3) can be regarded as a "viscous equation" of the equation (1.1), and define a weak solution. However, it found that it may not be the best "viscous equation" for other propose, see [10, 15]. Indeed, if one takes the equation (1.5) as our "viscous" approximation to (1.1), it is shown that it provide a way to compute the solution to (1.1) with a better rate of convergence. Second, these models also come from many physical situation, such as saturations theory.

However, there remain many open problems for such equation. Like behavior of solution for smooth periodic initial data. Is there a parallel theory as the non-periodic case? Although, mathematician already provided many numerical proof to show that the behavior is similar. However, it is still a good task for people to deal with this problem in a theoretical point of view.

Appendix A

Hopf-Cole transformation

The Hopf-Cole transformation is a useful tool to convert a special form of nonlinear equations into linear equations. It is especially useful to deal with a parabolic PDE with quadratic nonlinearity. Let's see how is this transformation powerful to convert nonlinear equations into linear equations, see [4].

Let's consider the initial-value problem for a quasi-linear parabolic equation,

$$\text{(A.1)} \qquad \begin{cases} u_t - a\Delta u + b|Du|^2 = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,t=0) = g(x) & \text{on} \quad \mathbb{R}^n, \end{cases}$$

where $a > 0$.

First, let's assume u is a smooth solution of (A.1), and set

$$\text{(A.2)} \qquad w := \phi(u),$$

where $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a smooth function; however, ϕ is not specified till this moment. Then we try to choose a suitable ϕ in order to find a transformation that w solves a linear equation.

Note that

$$\begin{cases} w_t &= \phi'(u)u_t \\ \Delta w &= \phi'(u)\Delta u + \phi''(u)|Du|^2, \end{cases}$$

and so with the equation (A.1), we have

$$\begin{aligned} w_t &= \phi'(u)u_t \\ &= \phi'(u)\left[a\Delta u - b|Du|^2\right] \\ &= a\Delta w - \left[a\phi''(u) + b\phi'(u)\right]|Du|^2, \end{aligned}$$

in order to force $w_t = a\Delta w$, choose ϕ satisfy $a\phi'' + b\phi' = 0$. And solve this ODE, and obtain

$$\phi = e^{-\frac{bu}{a}}.$$

For such setting, we can observe that if u solves (A.1), then

$$w = e^{-\frac{bu}{a}}. \tag{A.3}$$

And hence, we can transform our problem (A.1) into the following initial value problem in w,

$$\begin{cases} w_t - a\Delta w = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\ w(x, t = 0) = e^{-\frac{bg(x)}{a}} & \text{on} \quad \mathbb{R}^n. \end{cases} \tag{A.4}$$

Note that (A.3) is in fact our Hopf-Cole transformation.

By solving the problem (A.4), we have

$$w(x,t) = \frac{1}{(4\pi at)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4at}} e^{-\frac{b}{a}g(y)} dy \quad \text{where} \quad x \in \mathbb{R}^n, t > 0,$$

which is a unique bounded solution of (A.4).

And hence with (A.3), we obtain

$$\begin{aligned} u(x,t) &= -\frac{a}{b}\log w(x,t) \\ &= -\frac{a}{b}\log\left(\frac{1}{(4\pi at)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4at}}e^{-\frac{b}{a}g(y)}dy\right) \quad \text{where } x\in\mathbb{R}^n,\ t>0, \end{aligned}$$

which is the explicit solution of the quasi-linear initial value problem (A.1).

For this transformation, there is a very famous application for treating the Burger's equation in one dimension with a standard viscosity as follows:

$$u_t - \varepsilon u_{xx} + uu_x = 0 \quad \text{in } \mathbb{R}\times(0,\infty). \tag{A.5}$$

Although, (A.5) is not in the form of (A.1), one can set $u = w_x$ in (A.1) and integrate, then resulting to get

$$w_t - \varepsilon w_{xx} + \frac{1}{2}w_x^2 = 0 \quad \text{in } \mathbb{R}\times(0,\infty).$$

Now, it is in the form of (A.1), and we can apply the Hopf-Cole transformation to obtain a classical solution explicitly. And hence it yields a classical blow-up theory for the equation (A.5).

Appendix B

Dirichlet problem

In chapter 4, we have shown that for the problem (1.5), (1.2) with a smooth initial data which is large, the solution will blow-up in some finite time, T. And we also mention that it can also be shown by the numerical approach. However, for dealing a numerical problem, we cannot treat it without boundary, so it is important to introduce our blow-up result for a corresponding Dirichlet problem.

In this appendix, we deal with a problem almost the same as that we study in chapter 4. However, in this case, we study the problem with the boundary condition, so we are going to deal with a Dirichlet problem instead of a Cauchy problem. Let's consider the equation in a finite interval,

$$u_t + f(u)_x = \nu Q(u_x)_x \quad \text{in} \quad -L < x < L, \tag{B.1}$$

where $\nu > 0$; and with the boundary and initial conditions as:

$$u(x, t=0) = u_0(x) \quad \text{in} \quad -L < x < L; \tag{B.2}$$

$$u(\pm L, t) = u_0(\pm L) \quad \forall t \geq 0. \tag{B.3}$$

For this problem, we study its asymptotic behavior, and find that the blow-up behavior still holds. Indeed, we will extend the theorem 4.1 to a new theorem for

(B.1), (B.2) and (B.3).

Before we state the theorem, let's introduce the assumptions for this problem first. We assume that our problem with both the convective and dissipative fluxes satisfies the hypotheses (i)–(iv); and the initial data satisfies hypotheses (a) and (b). Furthermore, we impose the following two assumptions on the initial data instead of (c) and (d) in section 4.1.

(c') $u_0(L) \le u_0(x) \le 0 \quad \forall x \in [0, L]$;

(d') $u_0(x) \le \varphi(x) \quad \forall x \in [0, L]$.

However, with these assumptions and due to its skew symmetric property, (B.1), (B.2), (B.3) is equivalent to the following initial boundary value problem:

$$\text{(B.4)} \qquad \begin{cases} u_t + f(u)_x = \nu Q(u_x)_x, & 0 < x < L \\ u(x, t = 0) = u_0(x), & 0 < x < L \\ u(0, t) = 0, & \forall t \ge 0 \\ u(L, t) = u_0(L), & \forall t \ge 0. \end{cases}$$

Then we have the following generalization of the theorem 4.1.

THEOREM B.1 *Consider the initial boundary value problem (B.4) with all the hypotheses (i)–(iv), (a), (b) and (c'), (d') be satisfied. If it is further given that there exists a constant $u^* < 0$ such that $f(u^*) > 2Q_\infty$, and if $\varphi(x)$ is a smooth function satisfying $\varphi(L) = u^*$, and*

$$\text{(B.5)} \qquad Q(\varphi')_x - f(\varphi)_x < 0 \quad \forall x \in (0, L).$$

Then, there exists a finite $T > 0$ such that

$$\sup_{-L<x<L} |u_x(x,t)| \longrightarrow \infty \quad \text{as} \quad t \uparrow T.$$

Similar to the proof of theorem 4.1, we have to introduce some lemmas first. Now, let's state and prove some of the lemmas.

LEMMA B.2 *Suppose that the fluxes $f(u)$ and $Q(s)$ satisfy the hypotheses (i)–(iv). Then for any $u^* < 0$, there exists a constant $m > 1$, such that the boundary value problem*

$$\text{(B.6)} \qquad \begin{cases} f(\varphi)_x &= mQ(\varphi')_x \\ \varphi(0) &= 0 \\ \varphi(L) &= u^*. \end{cases}$$

has a solution.

Moreover, $\varphi''(x) > 0$, for all $x \in (0, L)$.

Proof Solving (B.6) gives

$$\text{(B.7)} \qquad \int_0^{\varphi(x)} \frac{du}{Q^{-1}\left(\frac{f(u)+c}{m}\right)} = x,$$

where c and m are constants. In order to satisfy the boundary condition, we need to choose c and m that satisfy

$$\int_0^{u^*} \frac{du}{Q^{-1}\left(\frac{f(u)+c}{m}\right)} = L.$$

Therefore, we need to show the existence of c and m.

Note that, by choosing an appropriate c and sufficiently large m, and due to the monotonicity of Q, $\varphi(x)$, defined by (B.7), will be bounded for all $0 \leq x < +\infty$. And hence, it is easy to observe that for $\bar{u} := \lim_{x \to +\infty} \varphi(x)$, $Q^{-1} \longrightarrow 0$, so we have $c = -f(\bar{u})$.

Set

$$G(\bar{u}, m) := \int_0^{u^*} \frac{du}{Q^{-1}\left(\frac{f(u)-f(\bar{u})}{m}\right)}.$$

Note that if we can show

$$G(\bar{u}, m) = L, \tag{B.8}$$

the proof is complete; however, observe that $G(\bar{u}, m)$ is a continuous function in both arguments, and that

$$G(\bar{u}, m) > 0 \ \forall \bar{u}, m \quad \text{and} \quad G(\bar{u}, m \longrightarrow +\infty) \longrightarrow +\infty. \tag{B.9}$$

So choosing $m = \frac{f(\bar{u})}{Q_\infty}$, then we have

$$\begin{aligned} G\left(\bar{u}, m = \frac{f(\bar{u})}{Q_\infty}\right) &= \int_{u^*}^{0} \frac{du}{Q^{-1}\left(Q_\infty\left(1 - \frac{f(u)}{f(\bar{u})}\right)\right)} \\ &\leq |u^*| \max_{u \in [u^*, 0]} \left\{ \frac{1}{Q^{-1}\left(Q_\infty\left(1 - \frac{f(u)}{f(\bar{u})}\right)\right)} \right\} \\ &= \frac{|u^*|}{Q^{-1}\left(Q_\infty\left(1 - \frac{f(u^*)}{f(\bar{u})}\right)\right)}. \end{aligned}$$

And hence, it is clearly that one can choose a sufficiently "large" $\bar{u}$, say $\bar{u} = \bar{u}_{min}$, then by the property of f, $Q^{-1}\left(Q_\infty\left(1 - \frac{f(u^*)}{f(\bar{u})}\right)\right)$ will be greater than $\frac{2|u^*|}{L}$, and consequently

$$G\left(\bar{u} = \bar{u}_{min}, m = \frac{f(\bar{u})}{Q_\infty}\right) \leq \frac{L}{2}. \tag{B.10}$$

And hence, by the continuity of G, (B.9) and (B.10) imply that there exist $\bar{u}$ and m such that (B.8) is satisfied. And so, the first statement of the lemma holds. That means that there exists φ satisfying (B.6).

Moreover, such φ also satisfies (4.4), (4.5). Hence by the lemma 4.2, we have $\varphi'' > 0$, and the proof is then completed. □

With lemma B.2, we can complete the proof of theorem B.1; the necessary steps are exactly identical to the proof of theorem 4.1, with the replacement of L instead of R; and so one can follow the arguments as before to show that the breakdown of the solution must holds. And hence the proof completed. Therefore, we conclude that the solution of the Dirichlet problem with a large initial boundary condition must blow-up in some finite time, T.

Remark In this appendix, we already extend the theorem 4.1 to the corresponding Dirichlet problem. On the other hand, it is found that such an extension also applies to the theorem 5.1, so that we have similar blow-up theorem for the problem in Dirichlet problem as in the Cauchy problem. Furthermore, the proof is similar to the proof of theorem 5.1, so it will be omitted here. For more details, the reader is refered to [6].

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